Equivariant K-theory, wreath products, and Heisenberg algebra

Weiqiang Wang*

Abstract

Given a finite group G and a G-space X, we show that a direct sum $\mathcal{F}_G(X) = \bigoplus_{n\geq 0} K_{G_n}(X^n) \bigotimes \mathbb{C}$ admits a natural graded Hopf algebra and λ-ring structure, where G_n denotes the wreath product $G \sim S_n$. $\mathcal{F}_G(X)$ is shown to be isomorphic to a certain supersymmetric product in terms of $K_G(X) \bigotimes \mathbb{C}$ as a graded algebra. We further prove that $\mathcal{F}_G(X)$ is isomorphic to the Fock space of an infinite dimensional Heisenberg (super)algebra. As one of several applications, we compute the orbifold Euler characteristic $e(X^n, G_n)$.

0 Introduction

Given a finite group G and a locally compact, Hausdorff, paracompact G-space X, the n-th direct product X^n admits a natural action of the wreath product $G_n = G \sim S_n$ which is a semi-direct product of the n-th direct product G^n of G and the symmetric group S_n. The main goal of the present paper is to study the equivariant topological K-theory $K_{G_n}(X^n)$, for all n together, and discuss several applications which are of independent interest.

We will first show that a direct sum

$$\mathcal{F}_G(X) = \bigoplus_{n\geq 0} K_{G_n}(X^n) \bigotimes \mathbb{C}$$

carries several wonderful structures. More explicitly, we show that $\mathcal{F}_G(X)$ admits a natural Hopf algebra structure with a certain induction functor as multiplication and a certain restriction functor as comultiplication (cf. Theorem 2). When X is a point, $K_{G_n}(X^n)$ is the Grothendieck ring $R(G_n)$, and we recover the standard Hopf algebra structure of $\bigoplus_{n\geq 0} R(G_n)$ (cf. e.g. [M1, M2, Z]). A key lemma used here is a straightforward generalization to equivariant K-theory of a statement in the representation theory of finite groups concerning the restriction of an induced representation to a subgroup.

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We show that $\mathcal{F}_G(X)$ is a free λ-ring generated by $K_G(X)\bigotimes\mathbb{C}$ (cf. Proposition 3). We write down explicitly the Adams operations φ^n's in $\mathcal{F}_G(X)$. Incidentally we also obtain an equivalent way of defining the Adams operations in $K_G(X)\bigotimes\mathbb{C}$ (not over $\mathbb{Z}$) by means of the wreath products, generalizing a definition by Atiyah [A] in terms of the symmetric group in the ordinary (i.e. non-equivariant) K-theory setting. When X is a point we recover the λ-ring structure of $\bigoplus_{n\geq 0} R(G_n)$ (cf. [M1]).

As a graded algebra $\mathcal{F}_G(X)$ has a simple description as a certain supersymmetric algebra in terms of $K_G(X)\bigotimes\mathbb{C}$ (cf. Theorem 3). The proof uses a theorem in [AS] and the structures of the centralizer group of an element in G_n and of the fixed point set of the action of $a\in G_n$ on X^n which we work out in Sect. 1. In particular, this description indicates that $\mathcal{F}_G(X)$ has the size of a Fock space of a certain infinite-dimensional Heisenberg superalgebra which we will construct in terms of natural additive maps in K-theory (cf. Theorem 4).

Our results above generalize Segal's work [S2], and our proofs are direct generalizations of those in [S2] (also see [Z, M1]). What Segal studied in [S2], partly motivated by remarks in Grojnowski [Gr], is the space $\bigoplus_{n\geq 0} K_{S_n}(X^n)\bigotimes\mathbb{C}$ for compact X which corresponds to our special case when G is trivial and then G_n is the symmetric group S_n. The paper [Gr] was in turn motivated by a physical paper of Vafa and Witten [VW]. Our present work grew out of an attempt to understand Segal's outlines [S2] and was also stimulated by Nakajima's lecture notes on Hilbert schemes [N] (also cf. [N1, Gr]). Our first main observation in this paper is that there is a natural way to add the group G into Segal's scheme and this allows several different applications as discussed below. These applications are of independent interest on their own. We expect that there is also a natural way to incorporate G into the remaining part of [S2].

Our addition of G has already highly non-trivial consequences even in the case when X is a point. By tensoring $\mathcal{F}_G(pt)$ with the group algebra of the Grothendieck ring $R(G)$, we obtain the underlying vector space for the vertex algebra associated to the lattice $R(G)$ [B, FLM]. When G is a finite subgroup of $SL_2(\mathbb{C})$, this will lead to a group theoretic construction of the Frenkel-Kac-Segal vertex representation of an affine Kac-Moody Lie algebra, which can be viewed as a new form of McKay correspondence [Mc]. Detail along these lines will be developed in a forthcoming paper.

An interesting case of our study is that X is the complex plane $\mathbb{C}^2$ acted upon by a finite subgroup G of $SL_2(\mathbb{C})$. Let $\widetilde{\mathbb{C}^2/G}$ denote the minimal resolution of singularities of $\mathbb{C}^2/G$ (cf. e.g. [N]). Via the McKay correspondence [Mc], we show that either $K_{G_n}((\mathbb{C}^2)^n)\otimes\mathbb{C}$ or $K_{S_n}((\widetilde{\mathbb{C}^2/G})^n)\otimes\mathbb{C}$ has the same dimension as the homology group of the Hilbert scheme of n points on $\widetilde{\mathbb{C}^2/G}$ (cf. [G]). This fact has a straightforward generalization, cf. Remark 6. Our message here is that the wreath product plays an important role in the study of the Hilbert scheme of n points on $\widetilde{\mathbb{C}^2/G}$ in exactly the way a symmetric group S_n does for the Hilbert

scheme of n points on $\mathbb{C}^2$ (cf. [N, BG]), which is in turn a special case of the former when G is trivial. We will discuss these in more detail in another occasion.

For a smooth manifold X acted upon by G, Dixon, Harvey, Vafa and Witten [DHVW] introduced a notion of orbifold Euler characteristic $e(X, G)$ in their study of string theory of orbifolds. We show that the orbifold Euler characteristic $e(X^n, G_n)$ is uniquely determined by $e(X, G)$ and n. In terms of a generating function, our formula reads (see Theorem 5):

$$\sum_{n \geq 1} e(X^n, G_n) q^n = \prod_{r=1}^{\infty} (1 - q^r)^{-e(X,G)}. \tag{1}$$

By putting $G = 1$ and thus $e(X, G) = e(X)$, we recover a formula of Hirzebruch-Höfer [HH]. By using Eq. (1) and Göttsche's formula [G], we show that X^n/G_n admits a resolution of singularities whose Euler characteristic coincides with the orbifold Euler characteristic $e(X^n, G_n)$ assuming that X is a smooth quasi-projective surface and X/G has a resolution of singularities whose Euler characteristic is $e(X, G)$.

In this paper the language of equivariant K-theory is used. We should also mention the very relevant construction of Heisenberg superalgebra on a direct sum over n of the homology group $H(X^{[n]})$ of Hilbert scheme of n points on a smooth quasi-projective surface X, due to Nakajima [N1] and Grojnowski [Gr] independently. However the constructions and computations in terms of K-theory are simpler and work for more general spaces. Bezrukavnikov and Ginzburg [BG] have proposed a way to obtain a direct isomorphism from $K_{S_n}(X^n) \bigotimes \mathbb{C}$ to $H(X^{[n]})$ for an algebraic surface X. Independently M. de Cataldo and L. Migliorini has recently established this isomorphism for complex surfaces [CM].

The plan of this paper is as follows. In Sect. 1 we give a presentation of the wreath product G_n and study its action on X^n. In Sect. 2 we construct a Hopf algebra structure on $\mathcal{F}_G(X)$. In Sect. 3 we give a description of $\mathcal{F}_G(X)$ as a graded algebra. In Sect. 4 we give a λ-ring structure on $\mathcal{F}_G(X)$. In Sect. 5 we construct the Heisenberg superalgebra which acts on $\mathcal{F}_G(X)$ irreducibly. In Sect. 6 we calculate the orbifold Euler characteristic $e(X^n, G_n)$ and study in detail the special case when X is the complex plane acted upon by a finite subgroup of $SL_2(\mathbb{C})$. We have included some detail which are probably trivial to experts hoping that this may benefit readers with different backgrounds.

1 The wreath product and its action on X^n

Let G be a finite group. We denote by G_* the set of conjugacy classes of G and $R(G)$ the Grothendieck ring of G. $R(G) \bigotimes_{\mathbb{Z}} \mathbb{C}$ can be identified with the ring of class functions $C(G)$ on G by taking the character of a representation. Denote by ζ_c the order of the centralizer of an element lying in the conjugacy class c in G.

We define an inner product on $C(G)$ as usual:

$$(\chi|\psi) = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi}(g), \quad \chi,\psi \in C(G). \tag{2}$$

Let $G^n = G\times\ldots\times G$ be the direct product of n copies of G. Denote by $|G|$ the order of G, and by $[g]$ the conjugacy class of $g\in G$. The symmetric group S_n acts on G^n by permuting the n factors: $s(g_1,\ldots,g_n) = (g_{s^{-1}(1)},\ldots,g_{s^{-1}(n)})$. The *wreath product* $G_n = G\sim S_n$ is defined to be the semidirect product of G^n and S_n, namely the multiplication on G_n is given by $(g,s)(h,t) = (g.s(h),st)$, where $g,h\in G^n, s,t\in S_n$. Note that G^n is a normal subgroup of G_n by identifying $g\in G^n$ with $(g,1)\in G_n$.

Given $a=(g,s)\in G_n$ where $g=(g_1,\ldots,g_n)$, we write $s\in S_n$ as a product of disjoint cycles: if $z=(i_1,\ldots,i_r)$ is one of them, the *cycle-product* $g_{i_r}g_{i_{r-1}}\ldots g_{i_1}$ of a corresponding to the cycle z is determined by g and z up to conjugacy. For each $c\in G_*$ and each integer $r\geq 1$, let $m_r(c)$ be the number of r-cycles in s whose cycle-product lies in c. Denote by $\rho(c)$ the partition having $m_r(c)$ parts equal to r $(r\geq 1)$ and denote by $\rho=(\rho(c))_{c\in G_*}$ the corresponding partition-valued function on G_*. Note that $||\rho|| := \sum_{c\in G_*}|\rho(c)| = \sum_{c\in G_*, r\geq 1} r m_r(c) = n$, where $|\rho(c)|$ is the size of the partition $\rho(c)$. Thus we have defined a map from G_n to $\mathcal{P}_n(G_*)$, the set of partition-valued function $\rho=(\rho(c))_{c\in G_*}$ on G_* such that $||\rho|| = n$. The function ρ is called the *type* of $a=(g,s)\in G_n$. Denote $\mathcal{P}(G_*) = \sum_{n\geq 0}\mathcal{P}_n(G_*)$.

Given a partition λ with m_r r-cycles $(r\geq 1)$, define $z_\lambda = \prod_{r\geq 1} r^{m_r} m_r!$. This is the order of the centralizer in S_n of an element of cycle-type λ. We shall denote by $l(\lambda) = \sum_{r\geq 1} m_r$ the length of λ.

Given a partition-valued function $\rho\in\mathcal{P}(G_*)$, we define $l(\rho) = \sum_{c\in G_*} l(\rho(c))$ and

$$Z_\rho = \prod_{c\in G_*} z_{\rho(c)}\zeta_c^{l(\rho(c))}.$$

Denote by $\sigma_n(c)$ the class function of G_n which takes value $n\zeta_c$ at an n-cycle whose cycle-product lies in $c\in G_*$ and 0 otherwise. For $\rho=\{m_r(c)\}_{c,r}\in\mathcal{P}(G_*)$, we define

$$\sigma^\rho = \prod_{c\in G_*, r\geq 1}\sigma_r(c)^{m_r(c)}.$$

We regard σ^ρ as the class function on G_n which takes value Z_ρ at elements of type ρ (where $n=||\rho||$) and 0 elsewhere.

We formulate some well-known facts below (cf. e.g. [M2]) which will be needed later.

Proposition 1 *Two elements in G_n are conjugate to each other if and only if they have the same type. The order of the centralizer in G_n of an element of type ρ is Z_ρ.*

We want to calculate the centralizer $Z_{G_n}(a)$ of $a \in G_n$. First we consider the typical case that a has one n-cycle. As the centralizers of conjugate elements are conjugate subgroups, we may assume that a is of the form $a = ((g, 1, \dots, 1), \tau)$ by Proposition 1, where $\tau = (12 \dots n)$. Denote by $Z_G^{\Delta}(g)$, or $Z_G^{\Delta_n}(g)$ when it is necessary to specify n, the following diagonal subgroup of G^n (and thus a subgroup of G_n):

$$Z_G^{\Delta}(g) = \{((h, \dots, h), 1) \in G^n \mid h \in Z_G(g)\}.$$

The following lemma follows from a direct computation.

Lemma 1 *The centralizer $Z_{G_n}(a)$ of a in G_n is equal to the product $Z_G^{\Delta}(g) \cdot \langle a \rangle$, where $\langle a \rangle$ is the cyclic subgroup of G_n generated by a. Moreover, $a^n \in Z_G^{\Delta}(g)$ and $|Z_{G_n}(a)| = n|Z_G(g)|$.*

Take a generic element $a = (g, s) \in G_n$ of type $\rho = (\rho(c))_{c \in G_*}$, where $\rho(c)$ has $m_r(c)$ r-cycles ($r \geq 1$). By Proposition 1, we may assume (by taking a conjugation if necessary) that the $m_r(c)$ r-cycles are of the form

$$g_{ur}(c) = ((g, 1, \dots, 1), (i_{u1}, \dots, i_{ur})), 1 \leq u \leq m_r(c), g \in c.$$

Denote $g_r(c) = ((g, 1, \dots, 1), (12 \dots r))$. Throughout the paper, $\prod_{c,r}$ is understood as the product $\prod_{c \in G_*, r \geq 1}$.

Lemma 2 *The centralizer $Z_{G_n}(a)$ of $a \in G_n$ is isomorphic to a direct product of the wreath products*

$$\prod_{c,r} \left(Z_{G_r}(g_r(c)) \sim S_{m_r(c)} \right). \tag{3}$$

Furthermore $Z_{G_r}(g_r(c))$ is isomorphic to $Z_G^{\Delta_r}(g) \cdot \langle g_r(c) \rangle$.

Proof. It follows from the first part of Lemma 1 that the centralizer $Z_{G_n}(a)$ should contain a certain subgroup naturally isomorphic to (3). By the second part of Lemma 1 we can count that the order of (3) is equal to Z_ρ. The lemma now follows by comparing with the order of $Z_{G_n}(a)$ given in Proposition 1. □

We will use $\star$ to denote the multiplication in $C(G_n)$ which corresponds to the tensor product in $R(G_n)$. We denote by $\underline{n}$ the trivial representation of G_n, and $\underline{1^n}$ the sign representation of G_n in which G^n acts trivially and S_n acts by ± 1 depending on a permutation is even or odd. By abuse of notations, we also use the same symbols to denote the corresponding characters as well. The following lemma follows easily from the definitions.

Lemma 3 *1) Given $\rho, \widetilde{\rho} \in \mathcal{P}_n(G_*)$, $\sigma^\rho \star \sigma^{\widetilde{\rho}} = \delta_{\rho, \widetilde{\rho}} Z_\rho \sigma^\rho$. In particular,*

$$\underline{n} = \sum_{||\rho||=n} Z_\rho^{-1} \sigma^\rho, \tag{4}$$

$$\underline{1^n} = \sum_{||\rho||=n} (-1)^{n-l(\rho)} Z_\rho^{-1} \sigma^\rho. \tag{5}$$

2) $\left(\sigma^\rho|\sigma^{\widetilde{\rho}}\right) = \delta_{\rho,\widetilde{\rho}} Z_\rho$. *In other words,* σ^ρ *takes value* Z_ρ *at the elements in* G_n *of type* ρ *and* 0 *elsewhere.*

For a G-space X, we define an action of G_n on X^n as follows: given $a = ((g_1, \ldots, g_n), s)$, we let

$$a.(x_1, \ldots, x_n) = (g_1 x_{s^{-1}(1)}, \ldots, g_n x_{s^{-1}(n)}) \tag{6}$$

where $x_1, \ldots, x_n \in X$. Next we want to determine the fixed point set $(X^n)^a$ for $a \in G_n$. Let us first calculate in the typical case $a = ((g, 1, \ldots, 1), \tau) \in G_n$. Note that the centralizer group $Z_G(g)$ preserves the g-fixed point set X^g.

Lemma 4 *The fixed point set is*

$$(X^n)^a = \{(x, \ldots, x) \in X^n \mid x = gx\}$$

which can be naturally identified with X^g. *The action of* $Z_{G_n}(a)$ *on* $(X^n)^a$ *can be identified canonically with that of* $Z_G(g)$ *on* X^g *together with the trivial action of the cyclic group* $\langle a \rangle$ *(cf. Lemma 1). Thus*

$$(X^n)^a / Z_{G_n}(a) \approx X^g / Z_G(g).$$

Proof. Let $(x_1, \ldots, x_n)$ be in the fixed point set $(X^n)^a$. By Eq. (6) we have

$$\begin{aligned} (x_1, x_2, x_3, \ldots, x_n) &= a.(x_1, x_2, x_3, \ldots, x_n) \\ &= (gx_n, x_1, x_2, \ldots, x_{n-1}). \end{aligned}$$

So all $x_i (i = 1, \ldots, n)$ are equal to, say x, and $gx = x$. The remaining statements follow from Lemma 1. □

All $Z_G(g)$ are conjugate and all X^g are homeomorphic to each other for different representatives g in a fixed conjugacy class $c \in G_*$. Also the orbit space $X^g/Z_G(g)$ can be identified with each other by conjugation for different representatives of g in $c \in G_*$. We make a convention to denote $Z_G(g)$ (resp. X^g, $X^g/Z_G(g)$) by $Z_G(c)$ (resp. X^c, $X^c/Z_G(c)$) by abuse of notations when the choice of a representative g in c is immaterial.

Lemma 5 *Retain the notations in Lemma 2. The fixed point set* $(X^n)^a$ *can be naturally identified with* $\prod_{c,r} (X^c)^{m_r(c)}$. *Furthermore the orbit space* $(X^n)^a / Z_{G_n}(a)$ *can be naturally identified with*

$$\prod_{c,r} S^{m_r(c)} \left(X^c / Z_G(c)\right)$$

where $S^m(\cdot)$ *denotes the* m*-th symmetric product.*

Proof. The first part easily follows from Lemma 4. By Lemma 2 and Lemma 4, the action of $Z_{G_n}(a)$ on $(X^n)^a$ can be naturally identified with that of

$$\prod_{c,r} \left(\left(Z_G^{\Delta_r}(g) \cdot \langle g_r(c) \rangle \right) \sim S_{m_r(c)} \right)$$

on $\prod_{c,r} (X^c)^{m_r(c)}$ where $S_{m_r(c)}$ acts by permutation and $\langle g_r(c) \rangle$ acts on X^c trivially. Thus the second part of the lemma follows. □

2 The Hopf algebra structure on $\mathcal{F}_G(X)$

Given a compact Hausdorff G-space X, we recall [A1, S1] that $K_G^0(X)$ is the Grothendieck group of G-vector bundles over X. One can define $K_G^1(X)$ in terms of the K^0 functor and a certain suspension operation, and one puts

$$K_G(X) = K_G^0(X) \bigoplus K_G^1(X).$$

The tensor product of vector bundles gives rise to a multiplication on $K_G(X)$ which is super (i.e. $\mathbb{Z}_2$-graded) commutative. In this paper we will be only concerned about the free part $K(X) \otimes \mathbb{C}$, which will be denoted by $\underline{K}_G(X)$ subsequently. We denote by $\dim K_G^i(X)(i = 0, 1)$ the dimension of $K_G^i(X) \otimes \mathbb{C}$.

If X is locally compact, Hausdorff and paracompact G-space, take the one-point compactification X^+ with the extra point ∞ fixed by G. Then we define $K_G^0(X)$ to be the kernel of the map

$$K_G^0(X^+) \longrightarrow K_G^0(\{\infty\})$$

induced by the inclusion map $\{\infty\} \hookrightarrow X^+$. It is clear that this definition is equivalent to the earlier one when X is compact. We also define $K_G^1(X) = K_G^1(X^+)$.

Note that $K_G(pt)$ is isomorphic to the Grothendieck ring $R(G)$ and $\underline{K}_G(pt)$ is isomorphic to the ring $C(G)$ of class functions on G. The bilinear map $\star$ induced from the tensor product

$$K_G(pt) \bigotimes K_G(X) \longrightarrow K_G(X)$$

gives rise to a natural $\underline{K}_G(pt)$-module structure on $\underline{K}_G(X)$. Thus $\underline{K}_G(X)$ naturally decomposes into a direct sum over the set of conjugacy classes G_* of G. The following theorem [AS] (also cf. [BC]) gives a description of each direct summand.

Theorem 1 *There is a natural $\mathbb{Z}_2$-graded isomorphism*

$$\phi : \underline{K}_G(X) \longrightarrow \bigoplus_{[g]} \underline{K}\left(X^g / Z_G(g)\right).$$

Given $c \in G_*$, we denote by σ_c the class function which takes value ζ_c at an element of G lying in the conjugacy class c and 0 otherwise. Then an element in $\underline{K}_G(X)$ can be written as of the form $\sum_{c \in G_*} \xi_c \sigma_c$, where $\xi_c \in \underline{K}\left(X^g / Z_G(g)\right)$. More explicitly the isomorphism ϕ is defined as follows when X is compact: if E is a G-vector bundle over X its restriction to X^g is acted by g with base points fixed. Thus $E|_{X^g}$ splits into a direct sum of subbundles E_μ consisting of eigenspaces of g fiberwise for each eigenvalue μ of g. $Z_G(g)$ acts on X^g and one may check that $\sum_\mu \mu E_\mu$ indeed lies in the $Z_G(g)$-invariant part of $\underline{K}^0(X^g)$. Put

$$\phi_g(E) = \sum_\mu \mu E_\mu \in \underline{K}^0\left(X^g / Z_G(g)\right) = \underline{K}^0\left(X^g\right)^{Z_G(g)}.$$

The isomorphism ϕ on the K^0 part is given by $\phi = \bigoplus_{[g]\in G_*} \phi_g$. Then one easily extends the isomorphism ϕ to K^1 as $K^1_G(X)$ can be identified with the kernel of the map from $K^0_G(X \times S^1)$ to $K^0_G(X)$ given by the inclusion of a point in S^1. When X is a point the isomorphism ϕ becomes the map from a representation of G to its character. By some standard arguments using compact pairs [A1], the isomorphism remains valid for a locally compact, Hausdorff and paracompact G-space. The following lemma is well known.

Lemma 6 *Given a finite group G, a subgroup H of G, and a G-space X. There is a natural* induction functor $Ind = Ind_H^G : K_H(X) \longrightarrow K_G(X)$. *In particular when X is a point, the functor Ind_H^G reduces to the familiar induction functor of representations.*

Proof. Note that there is a G-equivariant isomorphism

$$G \times_H X \xrightarrow{\text{Iso}} G/H \times X \tag{7}$$

by sending $(g, x) \in G \times_H X$ to (gH, gx). We remark that although both sides of (7) remain well-defined for a H-spcase X without a G-action, the map Iso makes sense only for a G-space X. Denote by $p : G \times_H X \longrightarrow X$ the composition of the projection $G/H \times X$ to X with the isomorphism (7). As is well known [S1], one has a natural isomorphism

$$K_H(X) \longrightarrow K_G(G \times_H X)$$

by sending an H-equivariant vector bundle V on X to the G-equivariant vector bundle $G\times_H V$. The composition $p\circ\pi$ of the projection $p : G\times_H X \longrightarrow X$ with the bundle map $\pi : G\times_H V \longrightarrow G\times_H X$ sends (g, v) to $(g, \pi(v))$. One easily check that this gives rise to a well-defined G-equivariant vector bundle on X, which induces the induction functor $\mathrm{Ind}_H^G : K_H(X) \longrightarrow K_G(X)$. □

We denote by Res_H^G (or Res_H, or even Res, if there is no ambiguity) the *restriction functor* from $K_G(X)$ to $K_H(X)$ by regarding a G-equivariant vector bundle as an H-equivariant vector bundle. Denote

$$\mathcal{F}_G(X) = \bigoplus_{n\geq 0} \underline{K}_{G_n}(X^n), \quad \mathcal{F}^q_G(X) = \bigoplus\nolimits_{n\geq 0} q^n \underline{K}_{G_n}(X^n)$$

where q is a formal variable counting the graded structure of $\mathcal{F}_G(X)$. We introduce the notion of q-dimension:

$$\dim_q \mathcal{F}_G(X) = \sum_{n\geq 0} q^n \dim K_{G_n}(X^n).$$

Define a multiplication $\cdot$ on $\mathcal{F}_G(X)$ by a composition of the induction map and the Künneth isomorphism k:

$$\underline{K}_{G_n}(X^n) \bigotimes \underline{K}_{G_m}(X^m) \xrightarrow{k} \underline{K}_{G_n\times G_m}(X^{n+m}) \xrightarrow{\text{Ind}} \underline{K}_{G_{n+m}}(X^{n+m}). \tag{8}$$

We denote by 1 the unit in $K_{G_0}(X^0) \cong \mathbb{C}$.

On the other hand we can define a comultiplication Δ on $\mathcal{F}_G(X)$, given by a composition of the inverse of the Künneth isomorphism and the restriction from G_n to $G_k \times G_{n-k}$:

$$\underline{K}_{G_n}(X^n) \longrightarrow \bigoplus_{m=0}^{n} \underline{K}_{G_m \times G_{n-m}}(X^n) \xrightarrow{k^{-1}} \bigoplus_{m=0}^{n} \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_{n-m}}(X^{n-m}).$$

We define the counit $\epsilon : \mathcal{F}_G(X) \longrightarrow \mathbb{C}$ by sending $\underline{K}_{G_n}(X^n)$ $(n > 0)$ to 0 and $1 \in K_{G_0}(X^0) \approx \mathbb{C}$ to 1. The antipode can be also easily defined (see Remark 2).

Theorem 2 *With various operations defined as above, $\mathcal{F}_G(X)$ is a graded Hopf algebra.*

To prove Theorem 2, we will need some preparation. Given two subgroups H and L of a finite group Γ and a Γ-space Y, and let V be an H-equivariant vector bundle on Y. We denote the action of H on V by ρ. Choose a set of representatives S of the double coset $H \backslash \Gamma / L$. $H_s = sHs^{-1} \cap L$ is a subgroup of L for $s \in S$. We denote by V_s the H_s-equivariant vector bundle on Y which is the same as V as a vector bundle and has the conjugated action

$$\rho^s(x) = \rho(s^{-1}xs), \quad x \in H_s. \tag{9}$$

Lemma 7 *$Res_L Ind_H^\Gamma V$ is isomorphic to the direct sum of the L-equivariant vector bundles $Ind_{H_s}^L V_s$ for all $s \in H \backslash \Gamma / L$.*

One easily shows that one can extend V in Lemma 7 to the whole $K_H(Y)$. In the case $Y = pt$, an H-equivariant vector bundle is just an H-module, and the induction and restriction functors become the more familiar ones in representation theory. In such a case, the above lemma is standard (cf. e.g. [Ser], Proposition 2.2). In view of our construction of the induction functor and the restriction functor, the proof of Lemma 7 is essentially the same as in the case $X = pt$ which we refer to [Ser] for a proof.

Proof of Theorem 2. We will show below that the comultiplication Δ is an algebra homomorphism. The other Hopf algebra axioms are easy to check.

We apply Lemma 7 to the case $Y = X^N$, $L = G_m \times G_n$, $H = G_l \times G_r$, and $\Gamma = G_N$, where $l + r = m + n = N$. In this case the double coset $H \backslash \Gamma / L$ is isomorphic to $(S_l \times S_r) \backslash S_N / (S_m \times S_n)$ since $G_N = G^N \cdot S_N$ and $G_l \times G_r = G^N \cdot (S_l \times S_r)$. Furthermore $(S_l \times S_r) \backslash S_N / (S_m \times S_n)$ is parametrized by the 2×2 matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{10}$$

satisfying

$$\begin{aligned} a_{ij} \in \mathbb{Z}_+, &\quad i, j = 1, 2, \\ a_{11} + a_{12} = m, &\quad a_{21} + a_{22} = n, \\ a_{11} + a_{21} = l, &\quad a_{12} + a_{22} = r. \end{aligned} \tag{11}$$

We denote by $\mathcal{M}$ the set of all the 2×2 matrices of the form (10) satisfying the conditions (11).

Given $E \in \underline{K}_{G_l}(X^l), F \in \underline{K}_{G_r}(X^r)$, we calculate by using Lemma 7 as follows:

$$\begin{aligned} \mathrm{Res}_{(m,n)}\mathrm{Ind}_{(l,r)}^{N}(E \boxtimes F) &= \bigoplus_{A \in \mathcal{M}} \mathrm{Ind}_{A}^{(m,n)}(1_{a_{11}} \otimes T^{(a_{12},a_{21})} \otimes 1_{a_{22}})(\mathrm{Res}_{A'}(E \boxtimes F)) \\ &= \bigoplus_{A \in \mathcal{M}} \mathrm{Ind}_{(a_{11},a_{12})}^{m} F_1 \boxtimes \mathrm{Ind}_{(a_{21},a_{22})}^{n} F_2. \end{aligned} \tag{12}$$

Here the superscript or subscript (a, b) is a short-hand notations for $G_a \times G_b$. 1_a stands for the identity operator from $\underline{K}_{G_a}(X^a)$ to itself, and $\mathrm{Ind}_{(a,b)}^{a+b}$ for the induction functor from $\underline{K}_{G_a \times G_b}(X^{a+b})$ to $\underline{K}_{G_{a+b}}(X^{a+b})$. $T^{(a,b)}$ denotes the canonical functor from $\underline{K}_{G_a}(X^a) \otimes \underline{K}_{G_b}(X^b)$ to $\underline{K}_{G_b}(X^b) \otimes \underline{K}_{G_a}(X^a)$ by switching the factors with an appropriate sign coming from the $\mathbb{Z}_2$ grading of K-theory. Given $A \in \mathcal{M}$ of the form (10), the A in the expressions Res_A, Ind_A etc stands for $G_A \equiv G_{a_{11}} \times G_{a_{12}} \times G_{a_{21}} \times G_{a_{22}}$ while A' for $G_{A'} \equiv G_{a_{11}} \times G_{a_{21}} \times G_{a_{12}} \times G_{a_{22}}$. We wrote $(1 \otimes T^{(a_{12},a_{21})} \otimes 1)(\mathrm{Res}_{A'}(E \boxtimes F))$ as $F_1 \boxtimes F_2$ instead of a direct sum of the form $F_1 \boxtimes F_2$ in order to simplify notations, with F_i $(i = 1, 2)$ the corresponding elements in $K_{G_{a_{i1}} \times G_{a_{i2}}}(X^{a_{11}+a_{12}})$.

Now it is straightforward to check that the statement that Δ is an algebra homomorphism is just a reformulation of the identity obtained by summing Eq. (12) over all possible (m, n) with $m + n = N$. □

3 A description of $\mathcal{F}_G(X)$ as a graded algebra

In this section, we give an explicit description of $\mathcal{F}_G(X)$ as a graded algebra which in particular tells us the dimension of $\underline{K}_{G_n}(X^n)$.

Theorem 3 *As a $(\mathbb{Z}_+ \times \mathbb{Z}_2)$-graded algebra $\mathcal{F}_G^q(X)$ is isomorphic to the supersymmetric algebra $\mathcal{S}\left(\bigoplus_{n \geq 1} q^n \underline{K}_G(X)\right)$. In particular,*

$$\dim_q \mathcal{F}_G(X) = \frac{\prod_{r \geq 1}(1 + q^r)^{\dim K_G^1(X)}}{\prod_{r \geq 1}(1 - q^r)^{\dim K_G^0(X)}}.$$

The supersymmetric algebra here is equal to the tensor product of the symmetric algebra $S\left(\bigoplus_{n \geq 1} q^n \underline{K}_G^0(X)\right)$ and the exterior algebra $\Lambda\left(\bigoplus_{n \geq 1} q^n \underline{K}_G^1(X)\right)$.

Proof. Take $a \in G_n$ of type $\rho = \{m_r(c)\}_{c,r}$ as in Sect. 1. By Lemma 2 and Lemma 5 and the Künneth formula, we have

$$\begin{aligned} K((X^n)^a/Z_{G_n}(a)) &\approx \bigotimes_{c\in G_*, r\geq 1} \left(\left(K(X^c)^{Z_G(c)}\right)^{\bigotimes m_r(c)}\right)^{S_{m_r(c)}} \\ &\approx \bigotimes_{c\in G_*, r\geq 1} \mathcal{S}^{m_r(c)}(K(X^c/Z_G(c))). \end{aligned} \tag{13}$$

Thus if we take a summation of Eq. (13) over all conjugacy classes of G_n and all over $n \geq 0$, we obtain:

$$\begin{aligned} \mathcal{F}_G^q(X) &\approx \bigoplus_{n\geq 0} \bigoplus_{\{m_r(c)\}_{c,r}\in\mathcal{P}_n(G_*)} q^n \bigotimes_{c,r} \mathcal{S}^{m_r(c)}(\underline{K}(X^c/Z_G(c))) \qquad \text{by Theorem 1,} \\ &= \bigoplus_{\{m_r(c)\}_{c,r}\in\mathcal{P}(G_*)} \bigotimes_{c,r} \mathcal{S}^{m_r(c)}\left(q^r \underline{K}(X^c/Z_G(c))\right) \\ &= \bigoplus_{\{m_r\}_r} \bigotimes_{r\geq 1} \mathcal{S}^{m_r} \left(\bigoplus_{c\in G_*} q^r \underline{K}(X^c/Z_G(c))\right) \text{ by letting } m_r = \textstyle\sum_{c\in G_*} m_r(c), \\ &= \bigoplus_{\{m_r\}_r} \bigotimes_{r\geq 1} \mathcal{S}^{m_r}(q^r \underline{K}_G(X)) \qquad \text{by Theorem 1,} \\ &= \mathcal{S}\left(\bigoplus_{r\geq 1} q^r \underline{K}_G(X)\right). \end{aligned}$$

The statement concerning $\dim_q \mathcal{F}_G(X)$ is an immediate consequence. □

Remark 1 1) Theorem 3 in the case when G is the trivial group (and so $G_n = S_n$) is due to Segal [S2]. Our proof is adapted from his to the wreath product setting.

2) If G acts on X freely, so does G^n on X^n. Then we have the isomorphism $K(X/G) \approx K_G(X)$. Note that G^n is a normal subgroup of the wreath product G_n and the quotient G_n/G^n is isomorphic to S_n. By Proposition 2.1 in Segal [S1] we see that

$$K_{G_n}(X^n) \approx K_{G_n/G^n}(X^n/G^n) = K_{S_n}((X/G)^n). \tag{14}$$

Therefore Theorem 3 follows from the special case $G = 1$ of Theorem 3 (applying to X/G) and Eq. (14).

3) When X is a point, $\mathcal{F}_G(pt) = \bigoplus_{n\geq 0} R(G_n)$, and $\sigma^\rho, \rho \in \mathcal{P}(G_*)$ form a linear basis for $\mathcal{F}_G(pt)$ (cf. [M2]). In particular,

$$\dim_q \mathcal{F}_G(pt) = \prod_{r\geq 1} (1-q^r)^{-|G_*|}.$$

4) One may reformulate Theorem 3 in terms of the de-localized equivariant cohomology [BBM, BC] via the Chern character.

Remark 2 The Hopf algebra defined in Sect. 2 can be identified via the isomorphism in Theorem 3 with the standard one on the supersymmetric algebra $\mathcal{S}\left(\bigoplus_{n\geq 1}\underline{K}_G(X)[n]\right)$ by showing the sets of primitive vectors correspond to each other. Here $\underline{K}_G(X)[n]$ denotes the n-th copy of $\underline{K}_G(X)$ (see Theorem 3). The antipode of the former space can be transfered via the isomorphism from the latter one.

4 The λ-ring structure on $\mathcal{F}_G(X)$

Let us denote by c_n the conjugacy class in G_n which has the type of an n-cycle and whose cycle product lies in the conjugacy class $c \in G_*$. We consider the following diagram of K-theory maps:

$$
\begin{array}{rl}
\underline{K}_G(X) & \stackrel{\boxtimes n}{\longrightarrow} \ \underline{K}_{G_n}(X^n) \stackrel{\phi_n}{\longrightarrow} \bigoplus_{[a]\in(G_n)_*} \underline{K}\left((X^n)^a/Z_{G_n}(a)\right) \\
 & \underset{\iota}{\overset{pr}{\rightleftarrows}} \ \bigoplus_{c\in G_*} \underline{K}\left((X^n)^{c_n}/Z_{G_n}(c_n)\right) \stackrel{\vartheta}{\longrightarrow} \bigoplus_{c\in G_*} \underline{K}\left(X^c/Z_G(c)\right) \\
 & \stackrel{\phi}{\longleftarrow} \ \underline{K}_G(X).
\end{array}
$$

Given a G-equivariant vector bundle V, we define a G_n-action on the n-th outer tensor product $V^{\boxtimes n}$ by letting

$$((g_1, \ldots, g_n), s).v_1 \otimes \ldots \otimes v_n = g_1 v_{s^{-1}(1)} \otimes \ldots \otimes g_n v_{s^{-1}(n)} \tag{15}$$

where $g_1, \ldots, g_n \in G, s \in S_n$. Clearly $V^{\boxtimes n}$ endowed with such a G_n action is a G_n-equivariant vector bundle over X^n. Sending V to $V^{\boxtimes n}$ gives rise to the K-theory map $\boxtimes n$. ϕ_n is the isomorphism in Theorem 1 when applying to the case X^n with the action of G_n. pr is the projection to the direct sum over the conjugacy classes of G_n which are of the type of an n-cycle while ι denotes the inclusion map. ϑ denotes the natural identification given by Lemma 5. Finally the last map ϕ is the isomorphism given in Theorem 1.

We shall now define a λ-ring structure on $\mathcal{F}_G(X)$. It suffices to define the Adams operations ϕ^n on $\mathcal{F}_G(X)$. We also introduce several other K-theory operations which will be needed later.

Definition 1 *We define the following composition maps:*

$$
\begin{aligned}
\psi^n &:= n\phi^{-1} \circ \vartheta \circ pr \circ \phi_n \circ \boxtimes n : \underline{K}_G(X) \longrightarrow \underline{K}_G(X), \\
\varphi^n &:= n\phi_n^{-1} \circ \iota \circ pr \circ \phi_n \circ \boxtimes n : \underline{K}_G(X) \longrightarrow \underline{K}_{G_n}(X^n), \\
ch_n &:= \phi^{-1} \circ \vartheta \circ pr \circ \phi_n : \underline{K}_{G_n}(X^n) \longrightarrow \underline{K}_G(X), \\
\omega_n &:= n\phi_n^{-1} \circ \iota \circ \vartheta^{-1} \circ \phi : \underline{K}_G(X) \longrightarrow \underline{K}_{G_n}(X^n).
\end{aligned}
$$

We list some properties of these K-theory maps whose proof is straightforward.

Proposition 2 *The following identities hold:*

$$ch_n \circ \omega_n = n\, Id, \quad \omega_n \circ \psi^n = n\varphi^n, \quad ch_n \circ \varphi^n = n\psi^n.$$

Recall that $\sigma_n(c)$ is the class function of G_n which takes value $n\zeta_c$ at an n-cycle whose cycle-product lies in $c \in G_*$ and 0 otherwise.

Lemma 8 *1)* $\varphi^n(V) = \sum_{c \in G_*} \zeta_c^{-1} V^{\boxtimes n} \star \sigma_n(c)$.

2) Both ψ^n and φ^n are additive K-theory maps.

Note that the order of the centralizer of an element lying in the conjugacy class c_n is equal to $n\zeta_c$. The first part of the above lemma now follows from the definition of φ^n and Lemma 3. The second part can be proved in exactly the same way as in the symmetric group case [A]. We record here only a useful formula in the proof: Given V, W two G-equivariant vector bundles, let $[V]$ denote the corresponding element of V in $K_G(X)$. (In general we use V itself to denote the corresponding element in $K_G(X)$ by abuse of notation). Then

$$([V] - [W])^{\boxtimes n} = \sum_{j=0}^{n} (-1)^j \mathrm{Ind}_{G_{n-j} \times G_j}^{G_n} [V^{\boxtimes (n-j)} \boxtimes W^{\boxtimes j}] \in K_{G_n}(X^n). \tag{16}$$

Here $V^{\boxtimes (n-j)}$ endows the standard G_{n-j}-action given by substituting n with $n-j$ in Eq. (15), and G_j acts on $W^{\boxtimes j}$ by the tensor product of the standard G_j-action with the sign representation of G_j.

ψ^n's are the Adams operations on $\underline{K}_G(X)$, giving rise to the λ-ring structure on $\underline{K}_G(X)$. Theorem 3 ensures us that $\mathcal{F}_G(X)$ as a λ-ring is free and generated by $\underline{K}_G(X)$.

Proposition 3 *$\mathcal{F}_G(X)$ is a free λ-ring generated by $K_G(X) \bigotimes \mathbb{C}$, with φ^n's as the Adams operations.*

Remark 3 If X is a point, then $K_{G_n}(pt) = R(G_n)$ and $\mathcal{F}_G(pt) = \bigoplus_{n \geq 0} R(G_n)$. Our result reduces to the fact that $\mathcal{F}_G(pt)$ is a free λ-ring generated by G_* [M1]. In the case when $G = 1$, the proposition was due to Segal [S2].

Denote by $\widehat{\mathcal{F}}_G^q(X)$ the completion of $\mathcal{F}_G^q(X)$ which allows formal infinite sums. Given $V \in \underline{K}_G(X)$, we introduce $H(V,q), E(V,q) \in \widehat{\mathcal{F}}_G^q(X)$ as follows:

$$\begin{aligned} H(V,q) &= \bigoplus_{n \geq 0} q^n V^{\boxtimes n}, \\ E(V,q) &= \bigoplus_{n \geq 0} q^n V^{\boxtimes n} \star \underline{1^n}. \end{aligned}$$

Proposition 4 *One can express $H(V,q)$ and $E(V,q)$ in terms of $\varphi^r(V)$ as follows:*

$$\begin{aligned} H(V,q) &= \exp\left(\sum_{r>0}\frac{1}{r}\varphi^r(V)q^r\right), \qquad (17) \\ E(V,-q) &= \exp\left(-\sum_{r>0}\frac{1}{r}\varphi^r(V)q^r\right). \end{aligned}$$

Proof. We shall prove Eq. (17) by using Eq. (4). The formula for $E(V,-q)$ can be similarly obtained by using Eq. (5).

$$\begin{aligned} H(V,q) &= \bigoplus_{n\geq 0} q^n V^{\boxtimes n} \star \underline{n} \\ &= \bigoplus_{n\geq 0} q^n V^{\boxtimes n} \star \left(\sum_{||\rho||=n} Z_\rho^{-1}\sigma^\rho\right) \\ &= \bigoplus_{n\geq 0}\sum_{||\rho||=n}\left(Z_\rho^{-1}q^n V^{\boxtimes n}\star\sigma^\rho\right) \\ &\overset{(\star)}{=} \prod_{c\in G_*, r\geq 1}\frac{1}{m_r(c)!}\left(\frac{1}{r}q^r\zeta_c^{-1}V^{\boxtimes r}\star\sigma_r(c)\right)^{m_r(c)} \\ &= \exp\left(\sum_{r\geq 1}\frac{1}{r}q^r\sum_{c\in G_*}\zeta_c^{-1}V^{\boxtimes r}\star\sigma_r(c)\right) \qquad \text{by Lemma 8,} \\ &= \exp\left(\sum_{r\geq 1}\frac{1}{r}q^r\varphi^r(V)\right). \end{aligned}$$

Here the equation $(\star)$ is understood by means of the multiplication in $\mathcal{F}_G(X)$ given by the composition (8). $\square$

Corollary 1 *The λ-operations λ^n on the λ ring $\mathcal{F}_G(X)$ sends $V \in \underline{K}_G(X)$ to $V^{\boxtimes n} \star \underline{1^n}$.*

Combining with the additivity of φ^r, we have the following.

Corollary 2 *The following equations hold for $V, W \in \underline{K}_G(X)$:*

$$\begin{aligned} H(-V,q) &= E(V,-q) \\ H(V\bigoplus W,q) &= H(V,q)H(W,q). \end{aligned}$$

5 $\mathcal{F}_G(X)$ and a Heisenberg superalgebra

We see from Theorem 3 that $\mathcal{F}_G(X)$ has the same size of the tensor product of the Fock space of an infinite-dimensional Heisenberg algebra of rank $\dim K_G^0(X)$ and

that of an infinite-dimensional Clifford algebra of rank $\dim K^1_G(X)$. It is our next step to actually construct such an Heisenberg/Clifford algebra. We will simply refer to as Heisenberg superalgebra from now on.

The dual of $\underline{K}_G(X)$, denoted by $\underline{K}_G(X)^*$, is naturally $\mathbb{Z}_2$-graded as identified with $\underline{K}^0_G(X)^* \bigoplus \underline{K}^1_G(X)^*$. Denote by $\langle \cdot, \cdot \rangle$ the pairing between $\underline{K}_G(X)^*$ and $\underline{K}_G(X)$. For any $n, m \geq 1$ and $\eta \in \underline{K}_G(X)^*$, we define an additive map

$$a_{-m}(\eta) : \underline{K}_{G_n}(X^n) \longrightarrow \underline{K}_{G_{n-m}}(X^{n-m}) \tag{18}$$

as the composition

$$\begin{aligned} \underline{K}_{G_n}(X^n) \quad &\stackrel{\text{Res}}{\longrightarrow} \quad \underline{K}_{G_m \times G_{n-m}}(X^n) \stackrel{k^{-1}}{\longrightarrow} \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_{n-m}}(X^{n-m}) \\ &\stackrel{ch_m \otimes 1}{\longrightarrow} \quad \underline{K}_G(X) \bigotimes \underline{K}_{G_{n-m}}(X^{n-m}) \stackrel{\eta \otimes 1}{\longrightarrow} \underline{K}_{G_{n-m}}(X^{n-m}). \end{aligned}$$

On the other hand, we define for any $m \geq 1$ and $V \in \underline{K}_G(X)$ an additive map

$$a_m(V) : \underline{K}_{G_{n-m}}(X^{n-m}) \longrightarrow \underline{K}_{G_n}(X^n) \tag{19}$$

as the composition

$$\begin{aligned} \underline{K}_{G_{n-m}}(X^{n-m}) \quad &\stackrel{\omega_m(V) \boxtimes \cdot}{\longrightarrow} \quad \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_{n-m}}(X^{n-m}) \\ &\stackrel{k}{\longrightarrow} \quad \underline{K}_{G_m \times G_{n-m}}(X^n) \stackrel{\text{Ind}}{\longrightarrow} \underline{K}_{G_n}(X^n). \end{aligned}$$

Let $\mathcal{H}$ be the linear span of the operators $a_{-m}(\eta), a_m(V), m \geq 1$, $\eta \in \underline{K}_G(X)^*$, $V \in \underline{K}_G(X)$. Clearly $\mathcal{H}$ admits a natural $\mathbb{Z}_2$-gradation induced from that on $\underline{K}_G(X)$ and $\underline{K}_G(X)^*$. Below we shall use $[\cdot, \cdot]$ to denote the supercommutator as well. It is understood that $[a, b]$ is the anti-commutator $ab + ba$ when $a, b \in \mathcal{H}$ are both odd elements according to the $\mathbb{Z}_2$-gradation.

Theorem 4 *When acting on $\mathcal{F}_G(X)$, $\mathcal{H}$ satisfies the Heisenberg superalgebra commutation relations, namely for $m, l \geq 1$, $\eta, \eta' \in \underline{K}_G(X)^*$, $V, W \in \underline{K}_G(X)$,*

$$[a_{-m}(\eta), a_l(V)] = l\delta_{m,l}\langle \eta, V \rangle, \tag{20}$$

$$[a_m(W), a_l(V)] = 0, \tag{21}$$

$$[a_{-m}(\eta), a_{-l}(\eta')] = 0. \tag{22}$$

Furthermore $\mathcal{F}_G(X)$ is an irreducible representation of the Heisenberg superalgebra.

Proof. We may assume that V, η are homogeneous, say of degree v and e where $v, e \in \{0, 1\}$, according to the $\mathbb{Z}_2$-grading of $\underline{K}_G(X)$ and its dual. We keep using the notations in the proof of Theorem 2.

Given $E \in \underline{K}_{G_r}(X^r)$, we first observe by the definitions (18) and (19) that $a_{-m}(\eta)a_l(V)E$ (resp. $(-1)^{ve}a_l(V)a_{-m}(\eta)E$) is given by the composition from the top to the bottom along the left (resp. right) side of the diagram below:

$$
\begin{array}{ccc}
 & \underline{K}_{G_r}(X^r) & \\
 & \downarrow \omega_l(V) \boxtimes \cdot & \\
 & \underline{K}_{G_l}(X^l) \bigotimes \underline{K}_{G_r}(X^r) & \\
Ind \swarrow & & \searrow 1 \otimes \mathrm{Res} \\
\underline{K}_{G_N}(X^N) & & \underline{K}_{G_l}(X^l) \bigotimes \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_{r-m}}(X^{r-m}) \\
\| & & \downarrow T^{(l,m)} \otimes 1 \\
\underline{K}_{G_N}(X^N) & & \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_l}(X^l) \bigotimes \underline{K}_{G_{r-m}}(X^{r-m}) \\
\mathrm{Res} \searrow & & P \swarrow \mathrm{Ind} \otimes 1 \\
 & \underline{K}_{G_m}(X^m) \bigotimes \underline{K}_{G_n}(X^n) & \\
 & \downarrow ch_m \otimes 1 & \\
 & \underline{K}_{G}(X) \bigotimes \underline{K}_{G_n}(X^n) & \\
 & \downarrow \eta \otimes 1 & \\
 & \underline{K}_{G_n}(X^n) &
\end{array}
$$

Here and below it is understood that when a negative integer appears in indices the corresponding term is 0. To simplify notations, we put Res (resp. Ind) instead of the composition $k^{-1} \circ \mathrm{Res}$ (resp. $\mathrm{Ind} \circ k$) in the above diagram and below.

We denote by $\mathcal{M}'$ the set of all the 2×2 matrices of the form (10) satisfying (11) except the following two matrices

$$
\begin{bmatrix} 0 & m \\ l & r-m \end{bmatrix}, \quad \begin{bmatrix} m & 0 \\ l-m & r \end{bmatrix}.
$$

As in the proof of Theorem 2, we apply Lemma 7 to the case $Y = X^N, \Gamma = G_N, H = G_l \times G_r$, and $L = G_m \times G_n$, where $l + r = m + n = N$.

$$
\begin{aligned}
& \mathrm{Res}_{(m,n)} \mathrm{Ind}^N_{(l,r)} (\omega_l(V) \boxtimes E) \\
= \; & \bigoplus_{A \in \mathcal{M}} \mathrm{Ind}^{(m,n)}_A (1_{a_{11}} \otimes T^{(a_{12},a_{21})} \otimes 1_{a_{22}}) (\mathrm{Res}_{A'}(\omega_l(V) \boxtimes E)) \\
= \; & \bigoplus_{A \in \mathcal{M}} \mathrm{Ind}^m_{(a_{11},a_{12})} F_1 \boxtimes \mathrm{Ind}^n_{(a_{21},a_{22})} F_2 \\
= \; & \bigoplus_{A \in \mathcal{M}'} \mathrm{Ind}^m_{(a_{11},a_{12})} F_1 \boxtimes \mathrm{Ind}^n_{(a_{21},a_{22})} F_2 \\
& \bigoplus F_1 \boxtimes \mathrm{Ind}^n_{(l,r-m)} F_2 \bigoplus F_1 \boxtimes \mathrm{Ind}^n_{(l-m,r)} F_2 \\
= \; & \bigoplus_{A \in \mathcal{M}'} \mathrm{Ind}^m_{(a_{11},a_{12})} F_1 \boxtimes \mathrm{Ind}^n_{(a_{21},a_{22})} F_2 \\
& \bigoplus (1_m \otimes \mathrm{Ind}^n_{(l,r-m)})(T^{(l,m)} \otimes 1_{l-m})(1_l \otimes \mathrm{Res}_{(m,r-m)})(\omega_l(V) \boxtimes E) \qquad (23) \\
& \bigoplus (1_m \otimes \mathrm{Ind}^n_{(l-m,r)}))(\mathrm{Res}_{(m,l-m)} \otimes 1_r)(\omega_l(V) \boxtimes E).
\end{aligned}
$$

We get 0 when applying the map $ch_m \otimes 1$ to $\mathrm{Ind}^m_{(a_{11},a_{12})} F_1 \boxtimes \mathrm{Ind}^n_{(a_{21},a_{22})} F_2$ for $A \in \mathcal{M}'$ in (23) by Lemma 3. When applying $(\eta \otimes 1) \circ (ch_m \otimes 1)$ to the second term of the r.h.s. of (23), we obtain $(-1)^{ve} a_l(V) a_{-m}(\eta) E$. When applying $ch_m \otimes 1$ to the third term of the r.h.s. of (23), we get 0 if $m \neq l$ by Lemma 3. In the case $m = l$, the third term of the r.h.s. of (23) is simply $\omega_l(V) \boxtimes E$. When applying $(\eta \otimes 1) \circ (ch_m \otimes 1)$ to it, we get $l\langle \eta, V \rangle E$ by Proposition 2. Putting all these pieces together, we have proved Eq. (20).

We may assume that W is homogeneous of degree $w \in \{0, 1\}$ according to the $\mathbb{Z}_2$-grading of $\underline{K}_G(X)$. Eq. (21) is a consequence of the transitivity of the induction functor:

$$\begin{aligned}
& \mathrm{Ind}^{G_{m+l+r}}_{G_m \times G_{l+r}} \left(\omega_m(W) \boxtimes \mathrm{Ind}^{G_{l+r}}_{G_l \times G_r} (\omega_l(V) \boxtimes E) \right) \\
= \ & \mathrm{Ind}^{G_{m+l+r}}_{G_m \times G_{l+r}} \left(\mathrm{Ind}^{G_m \times G_{l+r}}_{G_m \times G_l \times G_r} (\omega_m(W) \boxtimes \omega_l(V) \boxtimes E) \right) \\
= \ & \mathrm{Ind}^{G_{m+l+r}}_{G_m \times G_l \times G_r} (\omega_m(W) \boxtimes \omega_l(V) \boxtimes E) \\
= \ & (-1)^{vw} \mathrm{Ind}^{G_{m+l+r}}_{G_l \times G_{m+r}} \left(\mathrm{Ind}^{G_l \times G_{m+r}}_{G_l \times G_m \times G_r} (\omega_l(V) \boxtimes \omega_m(W) \boxtimes E) \right) \\
= \ & \mathrm{Ind}^{G_{m+l+r}}_{G_l \times G_{m+r}} \left(\omega_l(V) \boxtimes \mathrm{Ind}^{G_{m+r}}_{G_m \times G_r} (\omega_m(W) \boxtimes E) \right).
\end{aligned}$$

Similarly Eq. (22) is a consequence of the transitivity of the restriction functor. The irreducibility of $\mathcal{F}_G(X)$ as a representation of $\mathcal{H}$ follows immediately from the q-dimension formula for $\mathcal{F}_G(X)$ given in Theorem 3. $\square$

In the special case $G = 1$, the Heisenberg superalgebra was constructed by Segal [S2] which differs slightly from ours. Our proof follows his strategy of proof as well.

Remark 4 One may consider the enlarged space

$$V_G := \mathcal{F}_G(pt) \bigotimes \mathbb{C}[R(G)]$$

where $\mathbb{C}[R(G)]$ is the group algebra of the lattice $R(G)$. Note that V_G is the underlying space for a lattice vertex algebra [B, FLM]. In particular, when G is a finite subgroup of $SL_2(\mathbb{C})$, the space V_G is closely related to the Frenkel-Kac-Segal vertex representation of an affine Lie algebra. In this way, we are able to obtain a new link between the subgroups of $SL_2(\mathbb{C})$ and the affine Lie algebras widely known as the McKay correspondence. Connections among symmetric functions, $\mathcal{F}_G(pt)$, V_G, and vertex operators will be developed in a forthcoming paper.

More generally, one may consider

$$V_G(X) := \mathcal{F}_G(X) \bigotimes \mathbb{C}[K_G(X)] \tag{24}$$

when $K_G(X)$ is torsion-free, where $\mathbb{C}[K_G(X)]$ is the group algebra of the lattice $\mathbb{C}[K_G(X)]$ (if $K_G(X)$ is not torsion-free we replace $K_G(X)$ in (24) by the free part of $K_G(X)$ over $\mathbb{Z}$).

6 The orbifold Euler characteristic $e(X^n, G_n)$

In the study of string theory on orbifolds, Dixon *et al* [DHVW] came up with a notion of *orbifold Euler characteristic* defined as follows:

$$e(X,G) = \frac{1}{|G|} \sum_{g_1 g_2 = g_2 g_1} e(X^{g_1,g_2}),$$

where X is a smooth G-manifold. X^{g_1,g_2} denotes the common fixed point set of g_1 and g_2 and $e(\cdot)$ denotes the usual Euler characteristic. One easily shows [HH] that the orbifold Euler characteristic can be equivalently defined as

$$e(X,G) = \sum_{[g]\in G_*} e(X^g/Z_G(g)). \tag{25}$$

Denote by $X^{(n)}$ the n-th symmetric product of X. Recall that Macdonald's formula [M] relates $e(X^{(n)})$ to $e(X)$ as follows:

$$\sum_{n=0} e(X^{(n)}) q^n = (1-q)^{-e(X)}. \tag{26}$$

The following theorem relates the orbifold Euler characteristic $e(X^n, G_n)$ to $e(X,G)$.

Theorem 5 *We have* $\sum_{n\geq 0} e(X^n, G_n) q^n = \prod_{r=1}^{\infty} (1-q^r)^{-e(X,G)}$.

Proof. For an alternative proof see Remark 5 below. By the definition of the orbifold Euler characteristic, Lemmas 2 and Lemma 5, we have

$$\begin{aligned}
\sum_{n\geq 0} e(X^n, G_n) q^n &= \sum_{n\geq 0} \sum_{[a]\in (G_n)_*} e\left((X^n)^a / Z_{G_n}(a)\right) q^n && \text{by Eq. (25),} \\
&\overset{(A)}{=} \sum_{n\geq 0} \sum_{\sum_r r m_r(c) = n} \prod_{c\in G_*} e\left((X^c/Z_G(c))^{m_r(c)} \right) q^n \\
&= \prod_{c\in G_*} \prod_{r\geq 1} \left(\sum_{m_r(c)\geq 0} e\left((X^c/Z_G(c))^{m_r(c)} \right) q^{m_r(c)} \right) \\
&= \prod_{c\in G_*} \prod_{r\geq 1} (1-q^r)^{-e(X^c/Z_G(c))} && \text{by applying (26) to } X^c/Z_G(c), \\
&= \frac{1}{\prod_{r=0}^{\infty} (1-q^r)^{e(X,G)}} && \text{by Eq. (25).}
\end{aligned}$$

Here Eq. (A) follows from Lemma 2 and Lemma 5. □

In the case when G is trivial, we recover a formula given in [HH].

Remark 5 According to Atiyah and Segal [AS] the orbifold Euler characteristic can be calculated in terms of equivariant K-theory:

$$e(X,G) = \dim K_G^0(X) - \dim K_G^1(X). \tag{27}$$

Theorem 5 follows from Theorem 3 by applying Eq. (27) to $K_{G_n}(X^n)$.

One is interested [DHVW, HH] in finding a resolution of singularities

$$\widetilde{X/G} \longrightarrow X/G$$

with the property

$$e(X,G) = e(\widetilde{X/G}).$$

We assume that X is a smooth quasi-projective surface with such a property in the following discussions. Denote by $X^{[n]}$ the Hilbert scheme of n points on X. According to Göttsche [G], the Euler characteristic of $X^{[n]}$ is given by:

$$\sum_{n \geq 0} e(X^{[n]}) q^n = \prod_{r=0}^{\infty} (1 - q^r)^{-e(X)}. \tag{28}$$

We note that X^n/G_n is naturally identified with $(X/G)^n/S_n$. The following commutative diagram

$$\begin{array}{ccc} \widetilde{X/G}^{[n]} & \rightarrow & \left(\widetilde{X/G}\right)^n / S_n \\ \downarrow & & \downarrow \\ X^n/G_n & \equiv & (X/G)^n/S_n \end{array}$$

implies that the Hilbert scheme $\widetilde{X/G}^{[n]}$ is a resolution of singularity of X^n/G_n (indeed it is semismall). By applying Eq. (28) to $\widetilde{X/G}$ and comparing with Theorem 5, we have the following corollary.

Corollary 3 *Let X be a smooth quasi-projective surface and assume that there exists a smooth resolution $\widetilde{X/G}$ of singularities of the orbifold X/G such that $e(\widetilde{X/G}) = e(X,G)$. Then there exists a resolution of singularities of X^n/G_n given by $\widetilde{X/G}^{[n]}$ satisfying*

$$e(X^n, G_n) = e\left(\widetilde{X/G}^{[n]}\right).$$

The assumption of the existence of the resolution $\widetilde{X/G}$ of singularities of X/G above is necessary as this is the special case of X^n/G_n for $n = 1$. In the setting of Corollary 3, we conjecture that $\widetilde{X/G}^{[n]}$ is a *crepant* resolution of X^n/G_n provided that $\widetilde{X/G}$ is a crepant resolution of singularities of X/G.

We consider a special case in detail. Let X be the complex plane $\mathbb{C}^2$ acted upon by a finite subgroup G of $SL_2(\mathbb{C})$. Via the McKay correspondence [Mc], there is a one-to-one correspondence between the finite subgroups of $SL_2(\mathbb{C})$ and the Dynkin diagrams of simply-laced types A_n, D_n, E_6, E_7 and E_8. Let us denote by $\mathfrak{g}$ the simple Lie algebra corresponding to G. From the exact correspondence we know that the rank of $\mathfrak{g}$ is $|G_*| - 1$.

The quotient $\mathbb{C}^2/G$ has an isolated simple singularity at 0. There exists a minimal resolution $\widetilde{\mathbb{C}^2/G}$ of $\mathbb{C}^2/G$ well known as ALE-spaces (cf. e.g. [N]). It is known that the second homology group $H_2(\widetilde{\mathbb{C}^2/G}, \mathbb{Z})$ is isomorphic to the root lattice of $\mathfrak{g}$, cf. e.g. [N]. In particular

$$\dim K\left(\widetilde{\mathbb{C}^2/G}\right) = \dim H\left(\widetilde{\mathbb{C}^2/G}\right) = |G_*|. \tag{29}$$

So if we apply Theorem 3 to the case $\widetilde{\mathbb{C}^2/G}$ with a trivial group action, we have

$$\sum_{n\geq 0} \dim K_{S_n}\left(\widetilde{\mathbb{C}^2/G}^n\right) q^n = \prod_{r\geq 1}(1-q^r)^{-|G_*|}.$$

On the other hand, by the Thom isomorphism [S1] we have

$$K_{G_n}(X^n) \approx K_{G_n}(pt) = R(G_n).$$

It follows from the part 3) of Remark 1 and Eq. (29) that

$$\sum_{n\geq 0} \dim K_{G_n}\left(\mathbb{C}^{2n}\right) q^n = \prod_{r\geq 1}(1-q^r)^{-|G_*|}.$$

We can obtain another numerical coincidence from a somewhat different point of view as follows. For a general quasi-projective smooth surface Y, a well-known result of Fogarty says that the Hilbert scheme of n points on Y, denoted by $Y^{[n]}$, is a smooth $2n$ dimensional manifold. The Betti numbers of $Y^{[n]}$ were computed by Göttsche [G]. In particular, Göttsche's formula yields the dimension of the homology group of $Y^{[n]}$:

$$\sum_{n\geq 0} \dim H(Y^{[n]})q^n = \prod_{r\geq 1}(1-q^r)^{-\dim H(Y)}. \tag{30}$$

We apply Eq. (30) to the case $Y = \widetilde{\mathbb{C}^2/G}$. It follows from Eq. (29) that

$$\sum_{n\geq 0} \dim H(\widetilde{\mathbb{C}^2/G}^{[n]})q^n = \prod_{r\geq 1}(1-q^r)^{-|G_*|}.$$

Therefore we have proved the following.

Proposition 5 *The spaces $K_{G_n}(\mathbb{C}^{2n}) \bigotimes \mathbb{C}$, $K_{S_n}(\widetilde{\mathbb{C}^2/G}^n) \bigotimes \mathbb{C}$ and $H(\widetilde{\mathbb{C}^2/G}^{[n]})$ have the same dimension.*

Since the minimal resolution $\widetilde{\mathbb{C}^2/G}$ of $\mathbb{C}^2/G$ has no odd dimensional homology, we have (cf. [HH, N])

$$e(\widetilde{\mathbb{C}^2/G}) = \dim H(\widetilde{\mathbb{C}^2/G}) = e(\mathbb{C}^2, G) = |G_*|.$$

The following corollary is a special and important case of Corollary 3.

Corollary 4 *Let G be a finite subgroup of $SL_2(\mathbb{C})$ and $\widetilde{\mathbb{C}^2/G}$ be the minimal resolution of $\mathbb{C}^2/G$. Then the Hilbert scheme of n points on $\widetilde{\mathbb{C}^2/G}$ is a resolution of singularities of $\mathbb{C}^{2n}/G_n$ whose Euler characteristic is equal to the orbifold Euler characteristic $e(\mathbb{C}^{2n}, G_n)$.*

Remark 6 The fact that the (graded) dimension of $K_{S_n}(X^n)$ equals that of the homology group of the Hilbert scheme of n points of X for a more general surface X holds by the same argument as above. Bezrukavnikov and Ginzburg [BG] have proposed a way to establish a direct isomorphism between these two spaces for an algebraic surface X. M. de Cataldo and L. Migliorini has recently established an isomorphism independently for any complex surface [CM].

Proposition 5 can be generalized as follows. Assume that X is a quasi-projective surface acted upon by G and there exists a smooth resolution of singularities $\widetilde{X/G}$ of X/G such that the dimension of $K^i_G(X)$ $(i = 0, 1)$ equals that of the even (resp. odd) dimensional homology group of $\widetilde{X/G}$. Then we conclude that the dimension of $K_{G_n}(X^n)$ is equal to that of the homology group of the Hilbert scheme of n points on $\widetilde{X/G}$. We conjecture the existence of a natural isomorphism from $K_{G_n}(X^n) \bigotimes \mathbb{C}$ to $H(\widetilde{X/G}^{[n]})$, assuming the (necessary) existence of an isomorphism from $K_G(X) \bigotimes \mathbb{C}$ to $H(\widetilde{X/G})$ or $K(\widetilde{X/G}) \bigotimes \mathbb{C}$.

We believe that this is just a first indication of intriguing relations between the Hilbert scheme of n points on $\widetilde{X/G}$ and the wreath product G_n. We will elaborate on this in another occasion. When G is trivial, it reduces to well-known relations between the Hilbert scheme of n points and the symmetric group S_n (cf. [N, BG]).

Remark 7 Let us consider a special case of Corollary 4 by putting $G = \mathbb{Z}_2$. The wreath product G_n in this case is exactly the Weyl group of B_n or C_n. It is interesting to compare with a "Hilbert scheme" associated to a reductive group of type B_n (or C_n) defined in [BG].

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Note 1 added. The idea here of relating the representation rings of wreath products associated to finite groups of $SL_2(\mathbb{C})$ and vertex representations of affine Lie algebras has been fully developed in a paper "Vertex representations via finite groups and the McKay correspondence" by I. Frenkel, N. Jing and the author.

Note 2 added. The connections among Hilbert schemes, wreath products and K-theory have been developed in my recent paper "Hilbert schemes, wreath products, and the McKay correspondence".

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Department of Mathematics, Yale University, New Haven, CT 06520, USA
E-mail: wqwang@math.yale.edu